From endomorphisms to regular subgroups, (bi-)skew braces, ...

Andrea Caranti[ac], Lorenzo Stefanello[ls]

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[ac] Università degli Studi di Trento
[ls] Università degli Studi di Pisa
Italy

L. N. Childs
Fixed-point free endomorphisms and Hopf Galois structures
Proc. Amer. Math. Soc. **141** (2013), 1255–1265

Let G be a group, and $\varphi \in \mathrm{End}(G)$ be

1. fixed point free (${}^{\varphi}g = g$ only for $g = 1$), and
2. abelian (${}^{\varphi}G$ is abelian, or equivalently ${}^{\varphi}[G, G] = 1$).

Then the subset

$$N = \{\, h \mapsto g \cdot h \cdot ({}^{\varphi}g)^{-1} : g \in G \,\}$$

of $\mathrm{Perm}(G)$ is a regular subgroup of $\mathrm{Perm}(G)$ which is normalised by the image $\lambda(G)$ of the left regular representation of G.

Omaha 2020 & 2021

Alan Koch
Abelian maps, bi-skew braces, and opposite pairs of Hopf-Galois structures
https://arxiv.org/abs/2007.08967
to appear, *Proc. Amer. Math. Soc.*

A.C. and Lorenzo Stefanello
From endomorphisms to bi-skew braces, regular subgroups, normalising graphs, the Yang-Baxter equation, and Hopf Galois structures
https://arxiv.org/abs/2104.01582

Koch (2020)

Theorem (Koch)

Let ψ *be an* abelian *endomorphism of the group* G, *(*${}^{\psi}G$ *is abelian, or equivalently* ${}^{\psi}[G, G] = 1$*). Consider the maps*

$$\nu(g) : G \to G, \quad h \mapsto g \cdot ({}^{\psi}g)^{-1} \cdot h \cdot {}^{\psi}g.$$

Then $N = \{\, \nu(g) : g \in G \,\}$ *is a regular subgroup of* $\mathrm{Perm}(G)$

- *which normalises* $\lambda(G)$, *and*
- *is normalised by* $\lambda(G)$.

The associated skew brace $(G, \circ, \cdot)$ *is thus a bi-skew brace.*

L. N. Childs
Bi-skew braces and Hopf Galois structures
New York Journal of Mathematics **25** (2019), 574–588

Variation

Let ψ be an *abelian* endomorphism of the group G. Then the set $\left\{ h \mapsto g \cdot {}^{\psi g^{-1}}h : g \in G \right\}$ is a regular subgroup of $\mathrm{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$. (Here ${}^{x}h = x \cdot h \cdot x^{-1}$.)

Let ψ be an *arbitrary* endomorphism of the group G, and let $\varepsilon = \pm 1$. For $g \in G$, define $\nu(g) \in \mathrm{Perm}(G)$ by

$$\nu(g) : h \mapsto g \cdot {}^{\psi g^{\varepsilon}}h.$$

The set $N = \{ \nu(g) : g \in G \}$ is a *regular subset* of $\mathrm{Perm}(G)$, meaning that the map

$$N \to G, \qquad n \mapsto {}^{n}1$$

is a bijection, as ${}^{\nu(g)}1 = g$. Moreover, the set N normalises $\lambda(G)$.

1. When is N a subgroup?
2. When is N normalised by $\lambda(G)$?

$\varepsilon = -1$

Data: $N = \left\{ h \mapsto g \cdot {}^{\psi g^{-1}} h : g \in G \right\}$, a regular subset of $\mathrm{Perm}(G)$ which normalises $\lambda(G)$.

Theorem ($\varepsilon = -1$)

The following are equivalent:

1. *N is a subgroup of* $\mathrm{Perm}(G)$,
2. *N is a regular subgroup of* $\mathrm{Perm}(G)$ *which normalises* $\lambda(G)$,
3. *N is a regular subgroup of* $\mathrm{Perm}(G)$ *which normalises, and is normalised by,* $\lambda(G)$,
4. *the set N is normalised by* $\lambda(G)$,
5. ${}^{\psi}[[G, \psi], G] \leq Z(G)$, (abuse of notation: $[g, \psi] = g \cdot {}^{\psi}(g^{-1})$)
6. $(G, \cdot, \circ)$ *is a bi-skew brace, for* $g \circ h = g \cdot {}^{\psi g^{-1}} h$.

$\varepsilon = 1$, first part

Data: $N = \left\{ h \mapsto g \cdot {}^{\psi g}h : g \in G \right\}$, a regular subset of $\mathrm{Perm}(G)$ which normalises $\lambda(G)$.

Theorem ($\varepsilon = 1$, first part)

The following are equivalent:

1. *N is a subgroup of* $\mathrm{Perm}(G)$,
2. *N is a regular subgroup of* $\mathrm{Perm}(G)$ *which normalises* $\lambda(G)$,
3. ${}^{\psi}[{}^{\psi}G, G] \leq Z(G)$,
4. $(G, \cdot, \circ)$ *is a skew brace, for* $g \circ h = g \cdot {}^{\psi g}h$.

$\varepsilon = 1$, second part

Theorem ($\varepsilon = 1$, second part)

The following are equivalent:

1. *N is a regular subgroup of $\mathrm{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$,*
2. ${}^{\psi}[G, G] = [{}^{\psi}G, {}^{\psi}G] \le Z(G)$ *($\Rightarrow$ ${}^{\psi}G$ has nilpotence class ≤ 2),*
3. $(G, \cdot, \circ)$ *is a bi-skew brace, for $g \circ h = g \cdot {}^{\psi_g}h$.*

The condition ${}^{\psi}[G, G] \le Z(G)$ of (2)

- holds for all groups of nilpotence class ≤ 2 and all of their endomorphisms, and
- is implied by Koch's condition ${}^{\psi}[G, G] = 1$, and
- implies the conditions
  - ${}^{\psi}[[G, \psi], G] \le Z(G)$, and
  - ${}^{\psi}[{}^{\psi}G, G] \le Z(G)$

  of the previous results.

Koch (2021)

Alan Koch
Abelian maps, brace blocks, and solutions to the Yang-Baxter equation
https://arxiv.org/abs/2102.06104

Koch shows that if ψ is an abelian endomorphism of the group $(G, \cdot)$, then ψ is also an endomorphism of the group $(G, \circ)$, where

$$g \circ h = g \cdot {}^{\psi g^{-1}} h.$$

He is then able to iterate his construction, obtaining a sequence of group operations $\circ_n$ on G (starting with $\circ_0 = \cdot$). such that

$(G, \circ_n, \circ_m)$ is a (bi-)skew brace for each n, m.

Koch calls this a brace block.

Polynomials

Let G be any group, ψ be any endomorphism of G, and

$$p = a_n x^n + \cdots + a_1 x \in \mathbf{Z}[x]$$

be an integer polynomial with no costant term ($p(0) = 0$).

The map on G given by

$$g \mapsto {}^{p(\psi)}g = {}^{a_n\psi^n + \cdots + a_1\psi}g = {}^{\psi^n}g^{a_n} \cdot \dots \cdot {}^{\psi}g^{a_1}$$

is (not well defined and) not necessarily an endomorphism of G. For instance

$$\begin{aligned} {}^{\psi^2+\psi}(gh) &= {}^{\psi^2}(gh) \cdot {}^{\psi}(gh) = \\ &= {}^{\psi^2}g \cdot {}^{\psi^2}h \cdot {}^{\psi}g \cdot {}^{\psi}h = {}^{\psi^2}g \cdot {}^{\psi}g \cdot [{}^{\psi}g^{-1}, {}^{\psi^2}h] \cdot {}^{\psi^2}h \cdot {}^{\psi}h = \\ &= {}^{\psi^2+\psi}(g) \cdot [{}^{\psi}g^{-1}, {}^{\psi^2}h] \cdot {}^{\psi^2+\psi}(h), \end{aligned}$$

where $[a, b] = a \cdot b \cdot a^{-1} \cdot b^{-1}$.

Polynomials

We have

$$ {}^{\psi^2+\psi}(gh) = {}^{\psi^2+\psi}g \cdot [{}^{\psi}g^{-1}, {}^{\psi^2}h] \cdot {}^{\psi^2+\psi}h, $$

But if the condition

$$ {}^{\psi}[G, G] = [{}^{\psi}G, {}^{\psi}G] \leq Z(G) $$

holds, then

$$ {}^{({}^{\psi^2+\psi}(g \cdot h))}x = {}^{({}^{\psi^2+\psi}g) \cdot ({}^{\psi^2+\psi}h)}x. $$

In general, under the condition ${}^{\psi}[G, G] \leq Z(G)$,

- $x \mapsto {}^{({}^{p(\psi)}g)}x$ is well defined, and
- ${}^{({}^{p(\psi)}(g \cdot h))}x = {}^{({}^{p(\psi)}g) \cdot ({}^{p(\psi)}h)}x,$

for $p \in \mathbf{Z}[x]$, $p(0) = 0$, so that...

Brace Blocks and Normalising Graphs

Theorem (Brace Blocks a.k.a. Normalising Graphs)

Let $(G, \cdot) = (G, \circ_0)$ *be a group, and* $\psi \in \mathrm{End}(G, \cdot)$ *satisfying*

$$^{\psi}[G, G] \leq Z(G).$$

Let $p^{(n)} \in \{ p \in \mathbf{Z}[x] : p(0) = 0 \}$*, for* $n \geq 1$*.*

For each $n \geq 1$*, and* $g, h \in G$*, define recursively*

$$g \circ_n h = g \circ_{n-1} ({}^{p^{(n)}(\psi)}g \circ_{n-1} h \circ_{n-1} \overline{{}^{p^{(n)}(\psi)}g}),$$

where $\overline{\text{a bar}}$ *denotes the inverse with respect to* $\circ_{n-1}$*. Then*

$(G, \circ_n, \circ_m)$ *is a skew brace for all* $n, m \in \mathbf{N}$*.*

Skip Example

Skip to End

An example

Let G be the free group G in three generators g_0, g_1, g_2 in the variety of groups of nilpotence class at most two; here $[G, G]$ is the free abelian group on the three generators $[g_1, g_0]$, $[g_2, g_0]$, $[g_2, g_1]$.

Let ψ be the endomorphism of G defined by $g_0 \mapsto g_1 \mapsto g_2 \mapsto 1$. With $p^{(n)} = x$ for all n one can see that

$$\begin{aligned} g_0 \circ_n g_0 &\equiv g_0^2 \cdot [{}^{n\psi}g_0, g_0] \pmod{\langle [g_2, g_0], [g_2, g_1] \rangle} \\ &\equiv g_0^2 \cdot [g_1, g_0]^n \pmod{\langle [g_2, g_0], [g_2, g_1] \rangle}, \end{aligned}$$

Thus all operations $\circ_n$ are distinct, and we have an infinite brace block.

⏩ Skip to End

Regular subgroups and regular representations

Let $(G, 1)$ be a pointed set. Every regular subgroup $N \leq \mathrm{Perm}(G)$ determines a group operation $\circ_N$ on G, obtained by transport of structure from

$$N \to G, \qquad n \mapsto {}^{n}1.$$

For the inverse $\nu : G \to N$ of this map we have

$$^{\nu(g)}1 = g, \quad \text{and} \quad \nu(h \circ_N g) = \nu(h)\nu(g).$$

Therefore

$$^{\nu(h)}g = {}^{\nu(h)}({}^{\nu(g)}1) = {}^{\nu(h)\nu(g)}1 = {}^{\nu(h \circ_N g)}1 = h \circ_N g.$$

This is a converse to Cayley's Theorem.

Brace Blocks and Normalising Graphs

Consider the *normalising graph* of the pointed set $(G, 1)$. This is the undirected graph $\mathfrak{G}$, first considered by Tim Kohl,

- whose vertices are the regular subgroup $N \leq \mathrm{Perm}(G)$, and
- in which two vertices N, M are joined by an edge if N and M normalise each other.

Then a clique (complete subgraph) $\mathfrak{C}$ in $\mathfrak{G}$ corresponds to a brace block, where the skew braces are

$$(G, \circ_N, \circ_M), \quad \text{for } N, M \in \mathfrak{C}.$$

That's All, Thanks!